

Vibrations of Bells

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SUMMARY

The forms and vibrational modes of church bells, carillon bells and handbells are discussed and compared. When the principal modes are classified in groups according to the numbers and locations of the nodal circles and meridians, the vibrational behaviours of church bells and carillon bells show similarities to those of the much smaller handbells. The different timbres are traced to the differences in the vibrational behaviour and the resulting spectra of partial tones. A subjective strike note, heard in the sound of church bells and carillon bells, does not appear in handbell sound, but a prominent octave partial in handbell sound is radiated indirectly by the fundamental mode of vibration. Modal frequencies in all three types of bells can be fitted to a suitably modified form of Chladni's law. Large church bells and carillon bells follow a $1/f$ scaling law; small carillon bells scale as f^{-n} , where n is slightly less than 1. In handbell sets, n may vary from about $\frac{1}{2}$ in the larger bells to $\frac{1}{3}$ in the smallest bells.

1. INTRODUCTION

Bells have belonged to nearly every culture in history. Bells with clappers existed in the Near East before 1000 BC. A set of tuned bells, dating from

the fifth century BC, was recently discovered in the Chinese province of Hubei. Although they have served many functions, bells have always been closely associated with religious practice, especially with calling people to worship. Since at least the early 17th century, rings of six or more bells, tuned to notes of the diatonic scale, have been found in English church towers.

Bells developed as Western musical instruments in the 15th and 16th centuries when bell founders discovered how to tune their partials harmonically. The founders in the Low Countries, especially the Hemony brothers, took the lead in tuning bells, and their many fine carillons became a source of national pride, which they still are.

Handbells also date back at least several centuries BC, although tuned handbells of the present-day type were developed in England in the 18th century. One early use of handbells was to provide tower bellringers with a convenient means to practice change ringing. In more recent years, handbell choirs have become popular in schools and churches—some 2000 choirs are reported in the USA alone.

The evolution of bells and the history of acoustical investigations of bell sound are discussed in three recent books, all of which include extensive bibliographies.^{1,2,3}

Analysis of the rich sound of a bell reveals many components or partials, each associated with a different mode of vibration of the bell. The various partials in the sound of a church bell or carillon bell are given such descriptive names as 'hum', 'fundamental', 'tierce', 'quint', 'nominal' etc. The most prominent partials in the sound of a tuned bell, like those of most musical instruments, are *harmonics* of a fundamental.

In this paper we will make a comparison between church bells, carillon bells and handbells, and discuss some recent investigations into their acoustical properties. These distinctly different musical instruments have a number of common properties, but also some important differences.

2. THE FORMS OF BELLS

In Fig. 1 we show the measured profiles of typical modern examples of (a) a church bell, (b) a handbell, and (c) a small carillon bell. Large carillon bells closely resemble the corresponding church bell produced by the same founder. Profile details vary from one founder to another and have changed over the years. The evolution of modern bells from their primitive forebears remains something of a mystery, not least because of the relatively large numbers of founders who were in production in previous centuries but have left no modern successors. Most Western bells have axial symmetry.

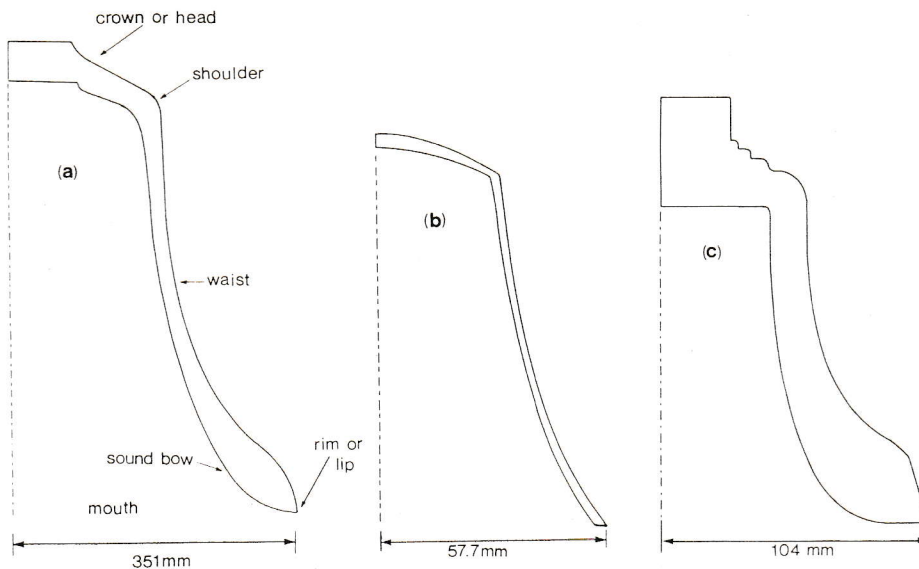


Fig. 1. Profiles of bells: (a) a church bell; (b) a handbell; (c) a small carillon bell.

Modern church and carillon bells have a thick soundbow where the metal clapper strikes, while most handbells have a more uniform thickness and clappers with a soft surface. The thinner walls of handbells make them considerably lighter and smaller than a church or carillon bell of the same pitch.

Most modern bells are cast of a bronze known as 'bell metal' with a composition of approximately 80% copper and 20% tin. The inside of a church bell or carillon bell is often turned down slightly on a lathe at various heights as individual partials are tuned, but the outside is left pretty much as cast. Most founders tune the first five partials, although it can be fewer with small carillon bells. Clearly, this tuning changes the profiles from the 'virgin' casting while retaining the axial symmetry, and this must be allowed for when analyzing the profiles of individual bells. Handbells are often cast considerably thicker than the final bell and turned down both inside and out, but they rarely have more than two partials tuned.

As Fig. 1 illustrates, the internal and external profiles of modern bells are not geometrically similar although the deviation from this is much less in handbells. For handbells, such as the $C_5^\dagger$ Malmk in Fig. 1(b), the thickness increases slowly from shoulder to rim. The situation is similar with most church bells, but with a greater rate of increase and a relatively

† Throughout this paper we use the ANSI recommended convention that C_4 denotes middle C (261.6 Hz).

thick ring of metal at the soundbow. The precise cross-sectional form of this ring varies considerably from one founder to another. A Taylor D_5 church bell is shown in Fig. 1(a) as a typical example. In the case of a small carillon bell, such as the C_6 Paccard shown in Fig. 1(c), the walls are much thicker, relative to its diameter, than are those for church bells and large carillon bells.

Even restricting our attention to a particular type of bell, say church bells, it is difficult to make general statements about profiles because each foundry has developed its own individual profiles, and these may have changed over the years. Two reasonably simple approaches may be described as 'circular arc' and 'elliptical arc', but some founders follow more complex procedures such as the use of various polynomials to describe their profiles.

In Fig. 2(a) we reproduce part of an 18th-century prescription for a church bell profile which is a typical example of the circular arc approach. Some modern versions of this in use in France and Italy are not very different. The head has inside and outside profiles which are circular arcs centered on the same point M on the symmetry axis while the region from shoulder to waist is described by circular arcs centered on the same point N. The section from the waist to the top of the soundbow is described by circular arcs centered on separate points P and Q, while the bottom of the soundbow is determined by a pair of straight lines.

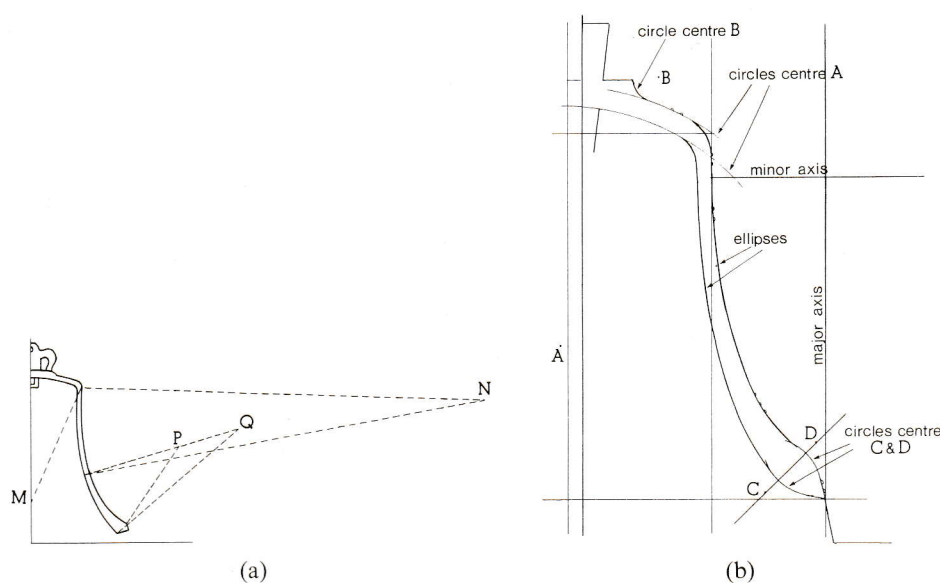


Fig. 2. (a) Church bell profile based on the circular arc approach. (b) Church bell profile based on the elliptical arc approach.

In Fig. 2(b) we show the profile of a Taylor bell typical of those based on the elliptical arc paradigm. The head is again determined by the arcs of circles at a common point on the axis of symmetry. However, the entire region from the shoulder to the top of the soundbow is now determined by one ellipse for the outside and another for the inside. These ellipses both have their major axes parallel to the symmetry axis and their centres slightly below the shoulder. The soundbow here is determined on the inside and outside by a pair of circular arcs which are mirror images of each other.

3. MODES OF VIBRATION

When struck by its clapper a bell is distorted and then vibrates in an exceedingly complex way. In principle, its vibrational motion can be described in terms of a linear combination of the normal modes of vibration whose initial amplitudes are determined by the form of the distortion. The complexity is due to the large number of normal modes of diverse character which contribute to the motion.

Group theoretical arguments predict that, in a bell with perfect axial symmetry, each mode should have a nodal pattern consisting of $2m$ meridians at equally spaced azimuths and n circles parallel to the rim, where $m, n = 0, 1, 2, \dots$.⁴ For $m = 0$ the modes are singlets, which can be described as 'breathing' modes and 'twisting' modes. Modes with $m > 0$ belong to degenerate doublets, with the nodal meridians of one member lying midway between those of the other but with their locations indeterminate until the symmetry is broken. The two members of such a doublet constitute a campanologist's 'partial'; for $m = 0$ a partial consists of a single mode.

In practice, the axial symmetry is broken by local variations in composition and wall thickness. This results in a splitting of the doublet which makes the timbre of the bell dependent, to some extent, on clapper orientation and also gives rise to the phenomenon of warble (see section 5.5). (The timbre is of course very dependent on the height of the clapper strike point even in bells with perfect axial symmetry.) Other complications can arise due to accidental degeneracy (i.e. degeneracy between modes other than the doublet pairs and so not a direct consequence of axial symmetry) producing mode mixing.

The acoustically important partials in a bell result from modes in which the motion is primarily normal to the bell's surface. It has been customary to classify these modes into families or groups with some common property of the nodal pattern for this component. Most important families have an

antinode for this component where the clapper strikes: at the soundbow for church bells and carillon bells, but a short distance above the lip for handbells.

3.1. Modes of a church bell

The first five modes of a church bell or carillon bell are shown in Fig. 3. Dashed lines indicate the locations of the nodes. The numbers (m, n) at the top denote the numbers of complete nodal meridians extending over the top of the bell (half the number of nodes observed along a circumference), and the numbers of nodal circles respectively. Note that

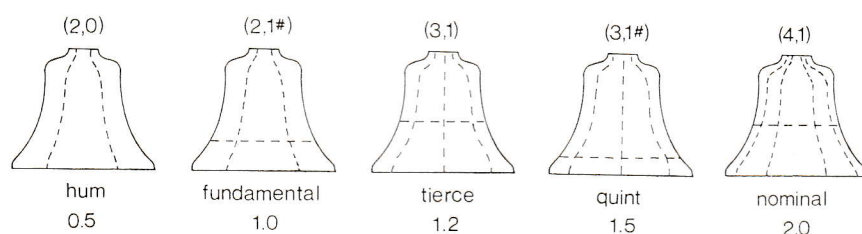


Fig. 3. The first five vibrational modes of a tuned church bell or carillon bell. Dashed lines indicate the nodes. Frequencies relative to the fundamental and names of the corresponding partials are given below each diagram.

there are two modes with $m = 3$ and $n = 1$, one with a circular node at the waist and one with a node near the sound bow. Thus, we follow the suggestion of Tyzzer⁵ and others and denote one as $(3, 1^\sharp)$ in Fig. 3 (although there is some question whether or not this is the best assignment of the $\sharp$ in view of family relationships with modes of large m). The ratio of each modal frequency to that of the prime is given at the bottom of the figure.

A detailed study of the vibrational modes of an English church bell has compared the normal modes computed by a finite element method to the first 134 modes observed in the laboratory.⁶ Modes such as $(2, 0)$, $(3, 1)$ and $(4, 1)$ are classified as 'ring driven' since, in the vicinity of the soundbow, they have many of the characteristics which the thick ring at the soundbow would exhibit if it were able to vibrate in its various inextensional radial modes as an independent system. These modes are sometimes referred to as 'group I modes'.⁷ They are excited strongly by the clapper, and they radiate most of the strong partials in the bell sound. The second important family, designated as 'group II', includes the $(2, 1^\sharp)$, $(3, 1^\sharp)$, $(4, 1^\sharp)$ and higher modes. They are classified as 'shell driven' modes, as are other important families having $n = 2, 3, 4, \dots$ and referred to as groups III, IV,

V, ... They are characterized by a nodal circle near the mouth. Like the 'ring driven' modes mentioned previously, they are inextensional in the sense that a neutral circle in each plane normal to the bell's symmetry axis remains unstretched. This means that the radial and tangential components of the motion, u and v respectively, are related by

$$u + \frac{\partial v}{\partial \theta} = 0$$

where θ is the polar angle in the plane concerned.⁸ Thus, employing also the θ part of the modal functions known from group theoretical considerations, we may write $u = m \sin m\theta$ and $v = \cos m\theta$. As m increases, these modes have radial components which become increasingly larger compared with their tangential ones. Examples are incorporated into Fig. 4.

The $(2, 1^\#)$ mode deserves further discussion. Its circular node in a D_5 church bell was observed to be 16 cm above the mouth, as compared with 29 cm in the $(3, 1)$ mode (the tierce or minor third) and 10 cm in the $(3, 1^\#)$ mode (the quint or fifth). Thus, insofar as positions of nodal circles are concerned, the mode fits into group II better than group I, but in a sense it serves both as the $(2, 1)$ and the $(2, 1^\#)$ modes in the mode classification scheme. Note that the $(2, 0)$ mode (hum) is the only normal mode in the

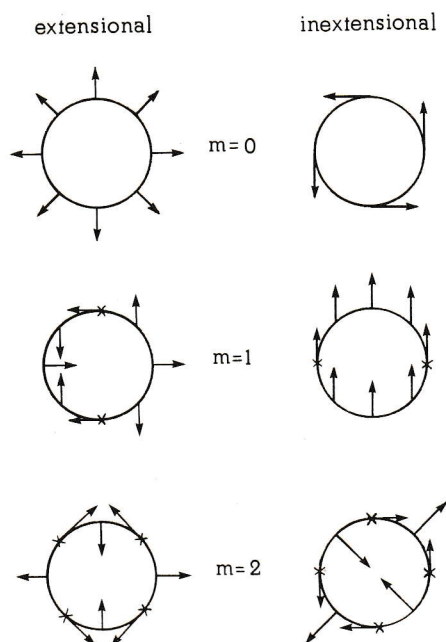


Fig. 4. Motion of a bell for modes of small m .

modern church bell without a circular node. In handbells there are two to three such modes, as we shall see, and experiments on a (somewhat conical) 14th-century church bell suggest that the same may be true in this bell also.⁹

Another family of ring driven modes, not much discussed in literature, are 'ring axial' modes in which the soundbow twists and moves in a longitudinal direction.⁶ A diagram based on a finite element calculation of the $m = 3$ case for a D_5 Taylor bell is shown in Fig. 5. Although such modes can occur quite low in the frequency spectrum (the $m = 2$ case occurs at 2.76 times the prime in this D_5 bell), they radiate very weakly as do the inextensional $m = 0$ 'twisting' and $m = 1$ 'swinging' modes and all the families of extensional modes. In the case of extensional modes, radial and tangential motions are related by

$$v + \frac{\partial u}{\partial \theta} = 0$$

so that $u = \cos m\theta$ and $v = m \sin m\theta$. Increasing m this time leads to increasing dominance of the tangential component over the radial. In Fig. 4 we show the forms of extensional and inextensional normal modes for $m = 0, 1$ and 2 as seen in an arbitrary plane normal to the symmetry axis.

It is possible to identify a number of families of extensional modes $\alpha, \beta, \gamma \dots$ where it is suggested that different segments of the bell (e.g. the soundbow in the case of α) go into their individual extensional modes and

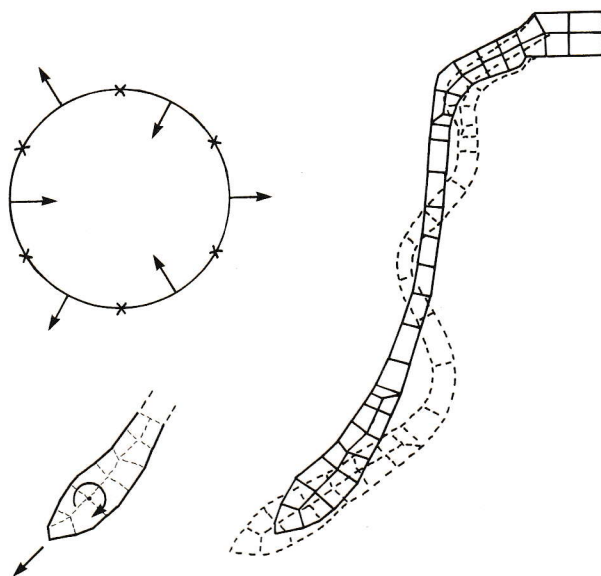


Fig. 5. Motion of a bell for $m = 3$, ring axial mode.

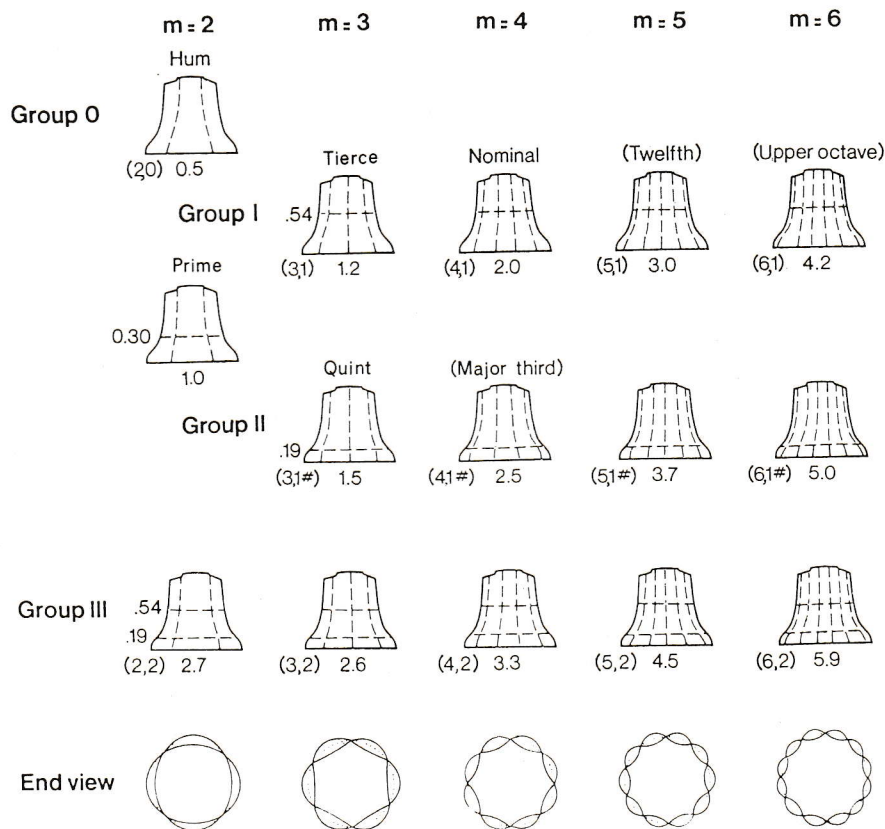


Fig. 6. Periodic table of inextensional modes of vibration in a church bell. Below each drawing are the modal frequencies of a D_5 church bell relative to the prime (which has essentially the same frequency as the strike note in a bell of high quality). At lower left (m, n) gives the number of nodal meridians $2m$ and nodal circles n .

act as drivers for the rest of the system. There is one member of each of these families for each m value. However, the situation is extremely complicated and further work is required to clarify it. Fortunately few, if any, of these modes are of any acoustical importance, except perhaps the lowest frequency cases for $m = 0$ and 1.

Figure 6 is a periodic table showing some of the modes observed in a D_5 church bell. The relative modal frequencies and the locations of the nodes are indicated. To the groups used by bell founders has been added a group 0 with a single member, the (2, 0) mode. This classification into groups makes it easier to compare church bell modes with handbell modes.

Vibrational frequencies of groups 0–IX in a church bell with a D_5 strike note are shown in Fig. 7. Also shown are the relative strengths at impact of several partials in the bell sound. Arrows denote the three partials in

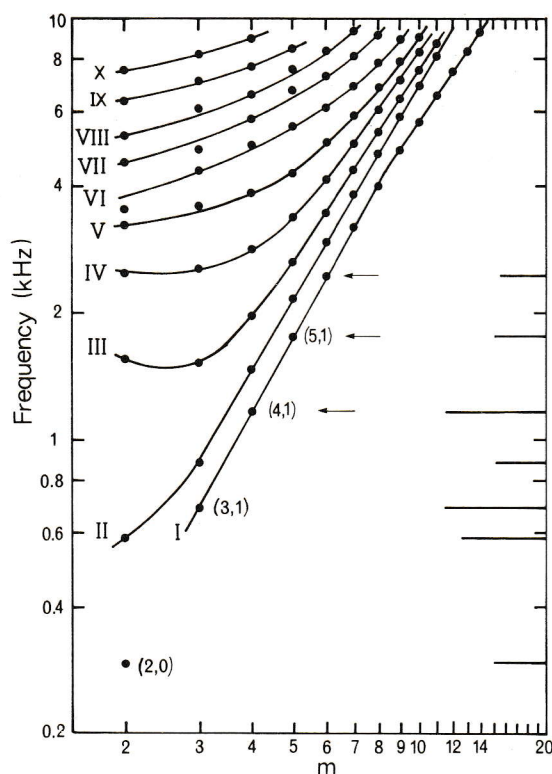


Fig. 7. Vibrational frequencies of groups 0–IX in a D_5 church bell (from Ref. 6). Also shown on the right are the relative strengths at impact of several partials in the bell sound. Arrows denote the three partials in group I that determine the strike note.

group I that determine the strike note (see section 5.2). Graphical displays of several modes computed by the finite element (PAFEC) method are shown in Fig. 8.

3.2. Modes of a handbell

Vibrational modes of a handbell are arranged in a periodic table in Fig. 9. Once again the numbers at lower left give (m, n) , the numbers of complete nodal meridians and nodal circles. Unlike those of a modern church bell, their order in frequency depends upon the size and shape of the handbell. The $(2, 0)$ and $(3, 0)$ modes are always the modes of lowest frequency. The next mode may be the $(3, 1)$, $(4, 0)$ or $(4, 1)$ depending upon the size of the handbell; the $(2, 1)$ mode occurs at a considerably higher frequency.¹⁰ The frequency order of the first eight modes in five Malmrk handbells is shown in Table 1.

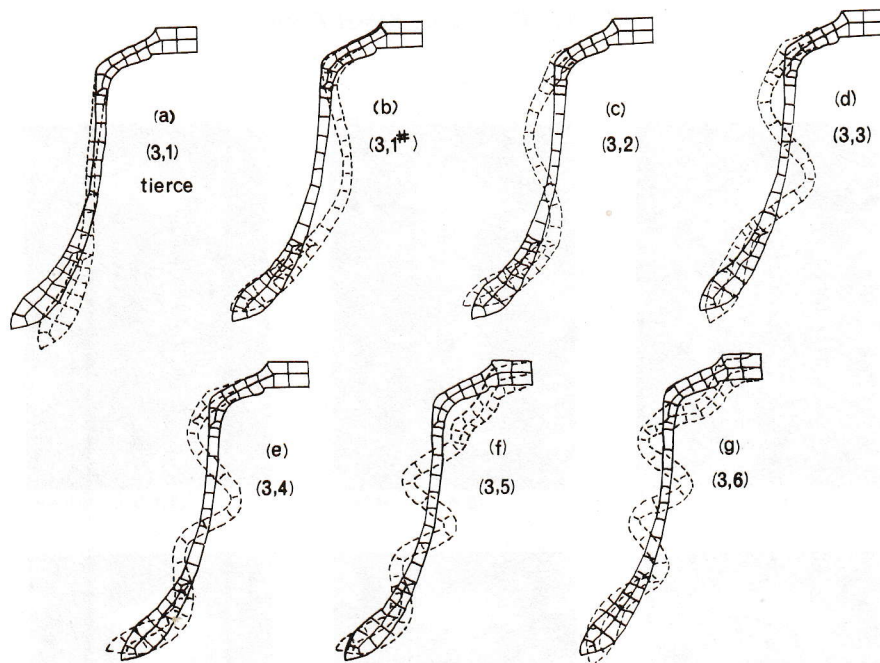


Fig. 8. Modal shapes in a church bell predicted by finite element calculation for the inextensional modes with $m=3$: (a) (3, 1); (b) (3, 1^{*}); (c) (3, 2); (d) (3, 3); (e) (3, 4); (f) (3, 5); (g) (3, 6) (from Ref. 6).

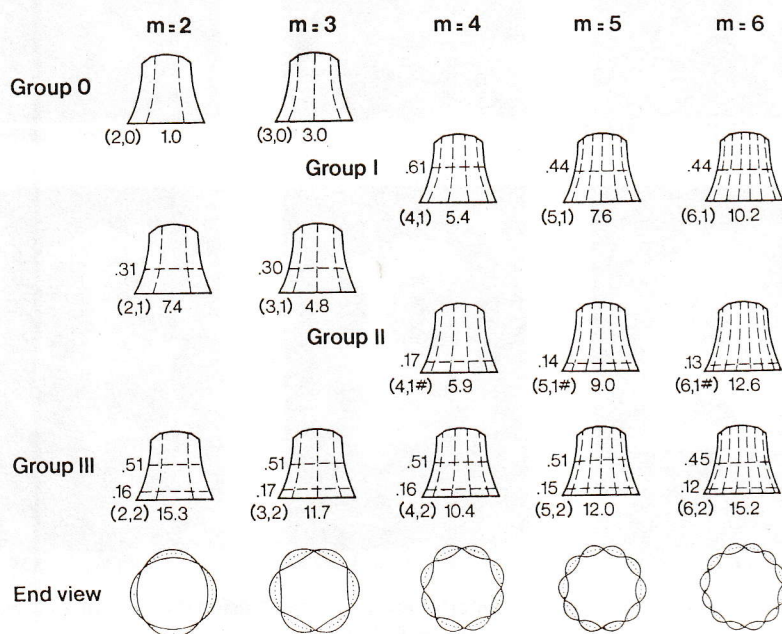


Fig. 9. Periodic table of inextensional modes in a handbell. Below each drawing are the relative modal frequencies in a Malmark C_5 handbell. At lower left (m, n) gives the number of nodal meridians $2m$ and nodal circles n .

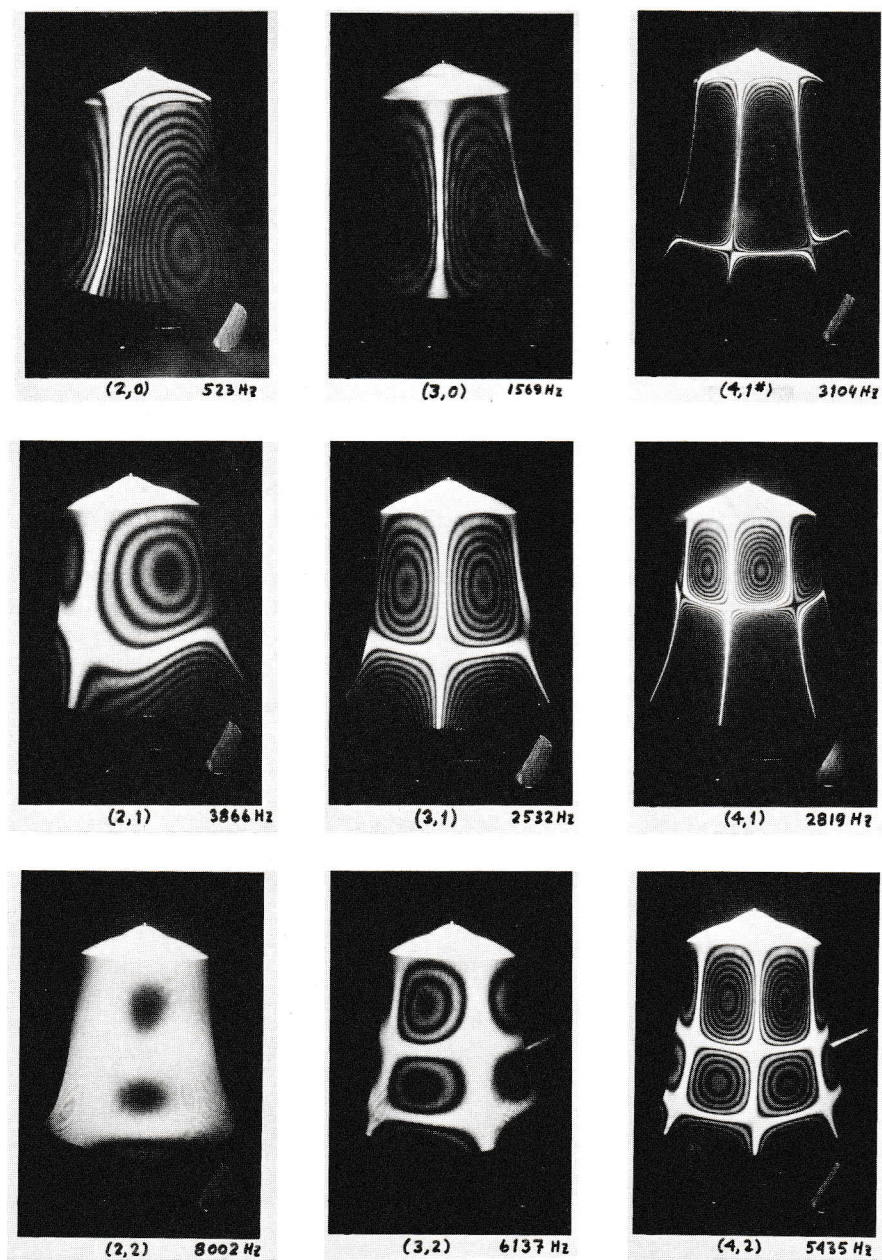


Fig. 10. Time-averaged hologram interferograms of inextensional modes in a C_5 handbell (from Ref. 33).

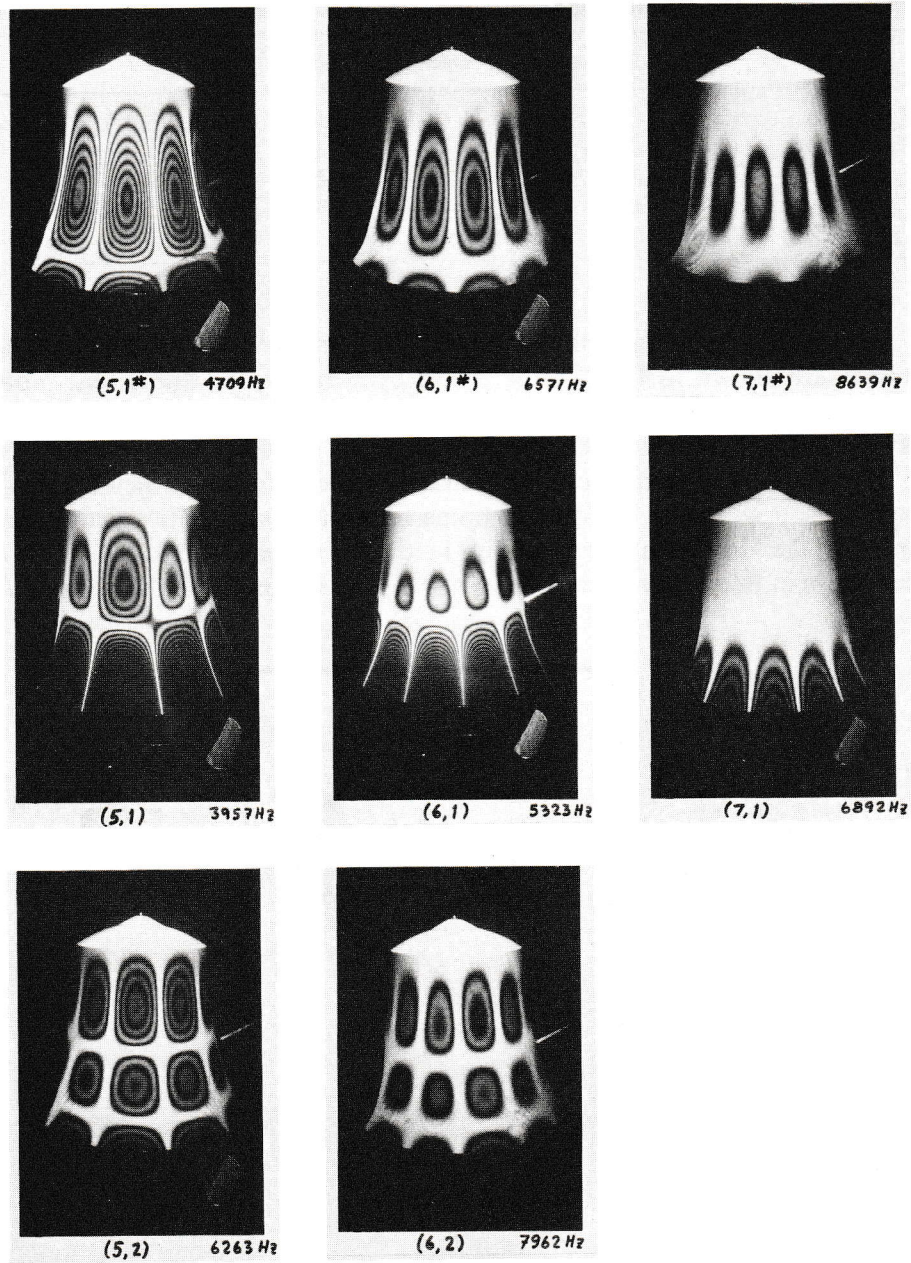
Fig. 10.—*contd.*

TABLE 1
Order in Frequency of Handbell Modes

Bell	Order in frequency							
G_2 (98 Hz)	(2, 0)	(3, 0)	(4, 0)	(4, 1)	(3, 1)	(5, 1)	(5, 1 [#])	(6, 1)
G_3 (192 Hz)	(2, 0)	(3, 0)	(4, 1)	(3, 1)	(4, 1 [#])	(5, 1)	(5, 1 [#])	(2, 1)
C_4 (261 Hz)	(2, 0)	(3, 0)	(3, 1)	(4, 1)	(4, 1 [#])	(5, 1)	(5, 1 [#])	(2, 1)
C_5 (523 Hz)	(2, 0)	(3, 0)	(3, 1)	(4, 1)	(4, 1 [#])	(2, 1)	(5, 1)	(5, 1 [#])
C_6 (1 047 Hz)	(2, 0)	(3, 0)	(3, 1)	(4, 1)	(2, 1)	(4, 1 [#])	(5, 1)	(5, 1 [#])

Hologram interferograms (see Powell and Stetson¹¹) of a number of the modes are shown in Fig. 10: the 'bull's eyes' locate the antinodes. Note that the upper half of the bell moves very little in the (7, 1) mode; the same is true in $(m, 1)$ modes when $m > 7$.

The vibrational frequencies of the C_5 handbell are shown in Fig. 11. A comparison with Fig. 7 shows up some interesting similarities and differences. Note that each curve in Fig. 11 drawn for $n = 1, 2, 3, \dots$ shows a minimum in frequency at about $m = n + 2$. This is similar to the behaviour

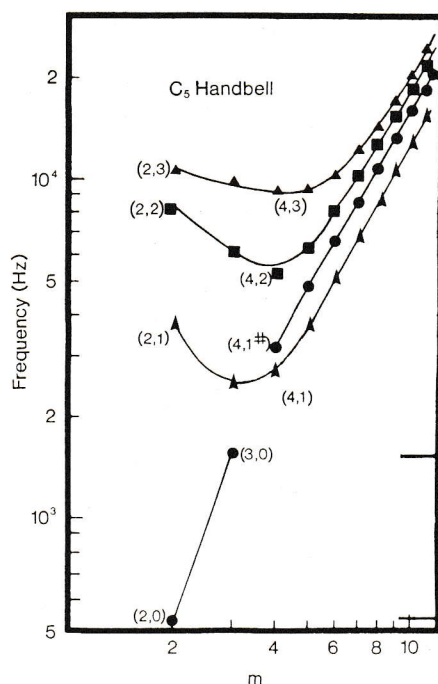


Fig. 11. Vibrational frequencies of a C_5 handbell (from Ref. 33). Also shown on the right are the relative strengths at impact of the two important partials in the bell sound.

of a cylinder with a fixed end cap, in which the stretching energy becomes substantial for small m .⁸ Thus the total strain due to stretching and bending in a cylindrical shell with fixed ends decreases with m until it reaches a minimum, then increases with m (see Fig. 7 in Arnold and Warburton¹²). In the larger G_2 and G_3 handbells, the minima occur at about $m = n + 3$.

The corresponding $n = 2, 3, 4, \dots$ curves in Fig. 7 tend to flatten out at small m , but they do not pass through a minimum (with the possible exception of $n = 2$). If one were determined to make Figs 7 and 11 as much alike as possible, one might connect the $(2, 1^\#)$ mode in Fig. 7 with the $(3, 1)$, $(4, 1)$, etc., modes and the $(2, 0)$ mode (hum) with the $(3, 1^\#)$, $(4, 1^\#)$, etc., modes; in Fig. 11 one would connect the $(2, 0)$ and $(3, 0)$ modes with the $(4, 1^\#)$, $(5, 1^\#)$, etc., modes. This is essentially the classification of church bell modes made by Grützmacher *et al.*¹³ and several others. It is less logical in terms of the physical shapes of the modes than the scheme used in Fig. 7, however. Note that the transition between $(m, 0)$ and $(m, 1)$ modes occurs at about the same m value ($m = 3$) for which the $n = 1$ curve goes through its minimum frequency. In the larger G_2 handbell (see Fig. 5 in Ref. 14) the minimum occurs at $m = 4$, and so does the transition between $(m, 0)$ and $(m, 1^\#)$. This phenomenon will be discussed further in a future publication.

4. CHLADNI'S LAW

Nearly two hundred years have passed since E. F. F. Chladni described his work on vibrating plates, including his well known method of sprinkling sand on plates to show the nodal lines.¹⁵ Chladni observed that the addition of one nodal circle raises the vibrational frequency of a flat circular plate by about the same amount as adding two nodal diameters, a relationship that is sometimes referred to as Chladni's law.⁸

Rayleigh gave this law a theoretical basis by showing analytically that for a flat circular plate with either free or clamped circumference the frequency takes the asymptotic form $f = c(m + 2n)^2$ in the limit of large $(m + 2n)$, where c is a constant for a given plate. In practice, it has been found that the frequencies of the vibrational modes in a wide variety of flat and non-flat circular plates can be fitted to a modified form of Chladni's law:

$$f_{m,n} = c_n(m + 2n)^{p_n}$$

In flat plates p_n is nearly 2, but in cymbals, bells and gongs it may vary from about 1.4 to 2.4.¹⁶

It can be shown that the modified Chladni's law (eqn 1) with $n = 0$ also holds for thin rings with $p = 2$ for inextensional and axial modes, and with $p = 1$ for extensional and torsional modes.¹⁷

4.1. Chladni's law applied to church bells

It is interesting to apply Chladni's law to bells of different kinds. For a 70 cm church bell the inextensional modes can be well fitted to curves of the type given by eqn (1), provided that different values of p_n and c_n are used for large and small m . For the ring-driven ($m, 1$) family of modes, $p_1 = 3.03$ for $m \leq 6$ and $p_1 = 1.85$ for $m > 6$, but for the shell-driven modes with $n \geq 4$, a smaller value of p_n must be used for small values of m , as shown in Table 2. In the families with $n = 4, 5$ and 6 , the change in p_n

TABLE 2
Shell-driven Church Bell Modes Fitted to Modified Chladni's Law $f_n = c_n(m + 2n)^{p_n}$
(Data from Ref. 17)

n	p_n	c_n (Hz)	m	p_n	c_n (Hz)	m
1				2.29	24	3-11
2				2.35	15	3-11
3				2.31	14	4-10
4	0.98	339	2-4	2.28	12	5-10
5	2.05	22	2-5	2.17	15	6-9
6	2.02	21	2-6	2.04	20	7-8
7	2.05	18	2-6			
8	1.70	48	2-5			
9	1.79	35	2-4			

occurs at $m = n$. Physically, it is noted that for $m > n$, little crown motion takes place. Also shown in Table 3 are the parameters required to fit various extensional modes to Chladni's law.¹⁷

An alternate modification of Chladni's law, which adds a parameter b , makes it possible to fit the various (m, n) families of inextensional modes using parameters which do not depend upon n .¹⁶

$$f = c(m + bn)^p \quad (2)$$

Again it is advantageous to select different values of p , c , and b for large and small m (i.e. for m above and below n in the families with $n = 4, 5$ and 6). The parameters used to fit the same church bell data to eqn (2) are given in Table 3. The best fit is obtained with $b = 0.81$ for large m ,

TABLE 3

Church Bell Modes Fitted to Modified Chladni's Law $f = c(m + bn)^p$ (Based on Experimental Data from Ref. 6)

Mode type	b	p	c (Hz)	m	b	p	c (Hz)	m
Ring-driven	0	1.92	79	2-6	0	1.54	162	7-14
inextensional	1	2.49	20	2-6	1	1.69	96	7-14
radial (RIR)	2	3.03	4.8	2-6	2	1.85	56	7-14
Shell-driven	1	1.40	24.2	$\leq n$	1	1.81	81	$> n$
$n = 1^2, 2, 3, \dots$	1.41	1.40	170	$\leq n$	0.81	1.81	89	$> n$
	2	1.37	130	$\leq n$	2	1.65	79	$> n$
Ring-driven axial (RA)	0	0.92	1 230	3-8				
Extensional	α	0	0.83	2-5				
	β	0	0.92	2-4				
	γ	0	0.55	2-3				
	δ	0	0.43	1-2				
	ϵ	0	0.18	2-3				

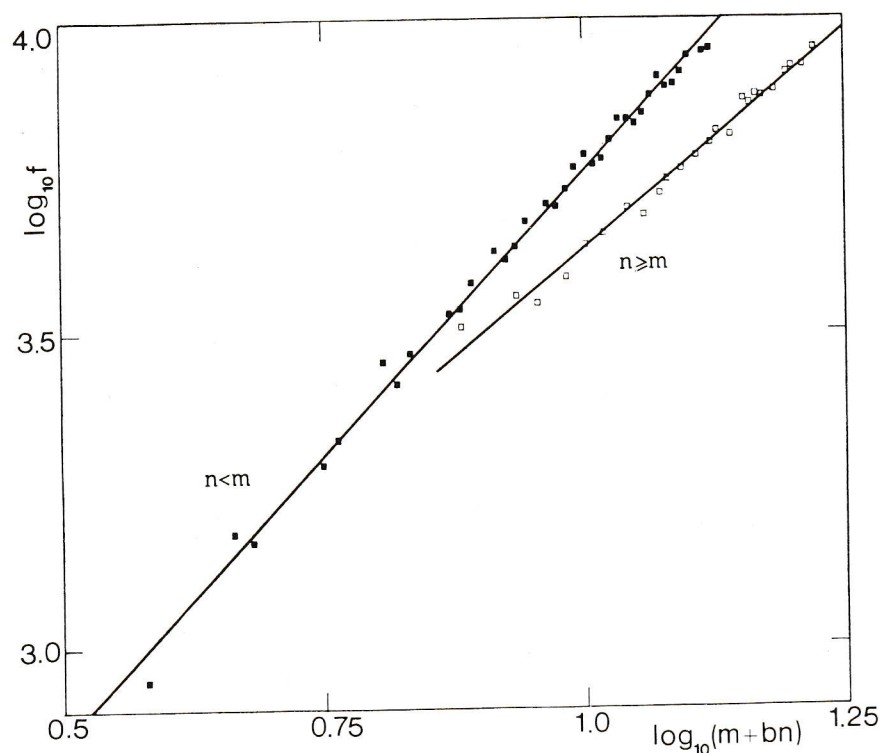


Fig. 12. Graph of f vs. $(m + bn)$ for a D_5 church bell with separate choices of b to optimize fits for $m \leq n$ and $m > n$ (from Ref. 17).

and $b = 1.41$ for small m for the shell-driven modes. However, using a value of $b = 1$ for all values of m gives nearly as good a fit.¹⁷ A decidedly poorer fit is obtained with $b = 2$. A graph for the optimum case is shown in Fig. 12.

4.2. Chladni's law applied to handbells

Figure 13 shows the frequencies of the modes in group 0 ($n = 0$), group I ($n = 1$) and group II ($n = 1^\sharp$) for seven Malmark handbells whose fundamental pitches cover a range of three and one-half octaves (G_2 to C_6 or 98–1047 Hz). As explained in section 3.2, the modes in group II have a minimum between $m = 3$ (for small bells) and $m = 4$ (for large bells) with this m value marking the top of the $(m, 0)$ and the bottom of the $(m, 1^\sharp)$ families.

Table 4 gives parameters used to fit the frequencies of three handbells to the modified Chladni's law (eqn 1). It is very likely that the data for each bell could be fitted to modified Chladni's law using a single set of constants (eqn 2) but this has not yet been done.

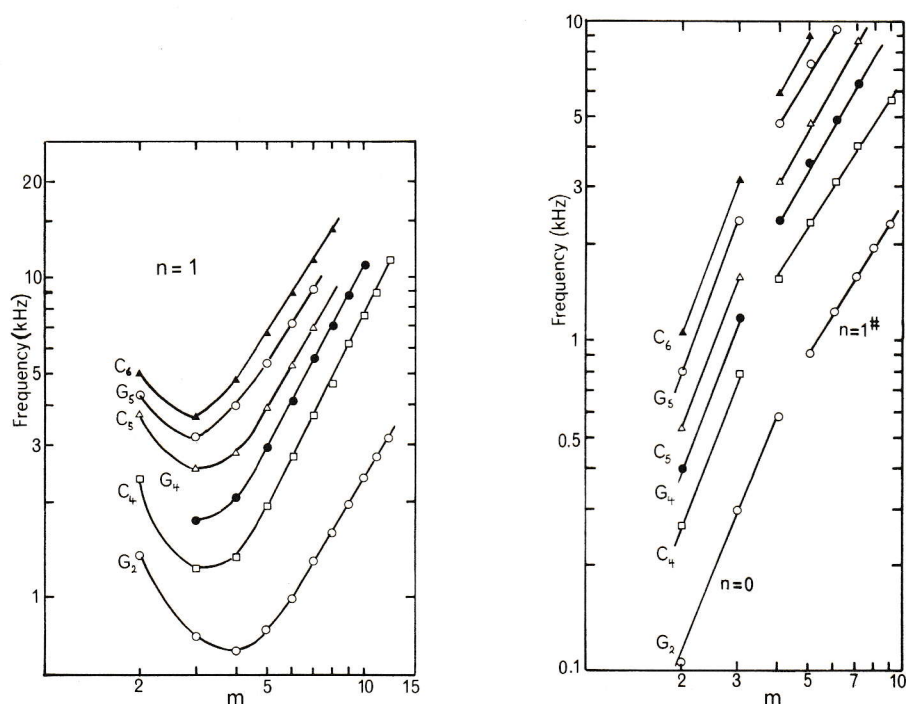


Fig. 13. Frequencies of $n=0$, $n=1^\sharp$, and $n=1$ modes in eight Malmark handbells covering a range of three and one-half octaves (G_2 to C_6) (adapted from Ref. 14).

TABLE 4
Handbells Fitted to Modified Chladni's Law $f = c_n(m + 2n)^{p_n}$

n	Bell	p_n	c_n (Hz)	Bell	p_n	c_n (Hz)	Bell	p_n	c_n (Hz)
0	G ₂	2.53	17.4	G ₃	2.68	29.8	C ₅	2.71	79.4
1 ²		2.04	17.0		2.29	50.3		2.53	33.9
1		2.14	11.5		2.37	14.1		2.20	53.7
2		1.81	26.9		2.36	14.6		1.70	15.5

5. TONAL CHARACTER

Although they are cast with the same bronze material and cover roughly the same range of pitch, the sounds of church bells, carillon bells, and handbells have distinctly different timbres. In a handbell, only two modes of vibration are tuned (although there are three harmonic partials in the sound), whereas in a church bell or large carillon bell at least five modes are tuned harmonically. As few as two modes are sometimes tuned in small carillon bells. A church bell or carillon bell is struck by a heavy metal clapper in order to radiate a sound that can be heard at a great distance, whereas the gentle sound of a handbell requires a relatively soft clapper.

5.1. Tuning

In the so-called English tuning of handbells, followed by most handbell makers in England and the USA, the (3, 0) mode is tuned to three times the frequency of the (2, 0) mode. The fundamental (2, 0) mode radiates a rather strong second harmonic partial, however, so that the sound spectrum has prominent partials at the first three harmonics.¹⁰ Some Dutch founders aim at tuning the (3, 0) mode in handbells to 2.4 times the frequency of the fundamental, giving their handbell sound a minor-third character somewhat like a church bell. Such bells are usually thicker and heavier than bells with the English-type tuning.¹⁸

Bell founders usually tune the lowest five modes of church bells and large carillon bells so that their vibrational frequencies are in the ratios 1:2:2.4:3:4. This is done by carefully thinning the inside of the bell at selected heights while it is mounted on a bell lathe. When this tuning is done, another five or six partials take on a nearly harmonic relationship, thus giving the bell a strong sense of pitch and a very musical quality. The various names of important partials are given in Table 5. Also given are the relative frequencies in an 'ideal bell' (just tuning) and a bell with partials tuned to equal temperament.

TABLE 5

Names and Relative Frequencies of Important Partial of a Tuned Church Bell or Carillon Bell

Mode	Names of partials	Note name	Ratio to prime (or strike note)			Decay time (s), bell in Fig. 7
			'Ideal' (just)	Equal temperament	Bell in Fig. 7	
(2, 0)	Hum, undertone	D ₄	0.500	0.500	0.500	52
(2, 1 [#])	Fundamental, prime	D ₅	1.000	1.000	1.000	16
(3, 1)	Tierce, minor third	F ₅	1.200	1.189	1.183	16
(3, 1 [#])	Quint, fifth	A ₅	1.500	1.498	1.506	6
(4, 1)	Nominal, octave	D ₆	2.000	2.000	2.000	3
(4, 1 [#])	Major third, deciem	F ₆	2.500	2.520	2.514	1.4
(2, 2)	Fourth, undeciem	G ₆	2.667	2.670	2.662	3.6
(5, 1)	Twelfth, duodeciem	A ₆	3.000	2.997	3.011	5
(6, 1)	Upper octave, double octave	D ₇	4.000	4.000	4.166	4.2
(7, 1)	Upper fourth, double undeciem	G ₇	5.333	5.339	5.433	3
(8, 1)	Upper sixth	B ₇	6.667	6.727	6.796	2
(9, 1)	Triple octave	D ₈	8.000	8.000	8.215	—

Notice that the highest four partials in Table 5 are raised by as much as 4% above those of the 'ideal' bell. This 'stretching' of the partial series may very well contribute a desirable quality to the bell sound.¹⁹

5.2. The strike note

When a large church bell or carillon bell is struck by its metal clapper, one first hears the sharp sound of metal on metal. This atonal strike sound includes many inharmonic partials that die out quickly, giving way to a strike note or strike tone that is dominated by the prominent partials of the bell. Most observers identify the metallic strike note as having a pitch at or near the frequency of the strong second partial (prime or fundamental), but to others its pitch is an octave higher (nominal). Finally, as the sound of the bell ebbs, the slowly decaying hum tone (an octave below the prime) lingers on. (For a historical account of investigation on the strike note, see Ref. 2, Part IV.)

The strike note is of great interest to psychoacousticians, because it is a subjective tone created by three strong nearly-harmonic partials in the bell sound. The nominal, the twelfth and the upper octave normally have frequencies nearly in the ratios 2:3:4 (see Table 5). The ear assumes these to be partials of a 'missing fundamental', which it hears as the strike note, or perhaps we should say, as the primary strike note.

In very large bells, a secondary strike note may occur a musical fourth above the primary strike note and may even appear louder under some conditions.²⁰ This secondary strike note is a subjective tone created by four partials beginning with the the upper octave. These partials are from the (6, 1), (7, 1), (8, 1) and (9, 1) modes of vibration, whose frequencies are nearly three, four, five and six times that of the secondary strike note (see Table 5). In a large bell (800 kg or more), these partials lie below 3000 Hz, where the residue pitch is quite strong.²¹

In small bells, the higher partials lie at a very high frequency where the residue pitch is weak, and so both the primary and secondary strike notes are weak. The pitch is then determined mainly by the hum, the prime, and the nominal, which normally are tuned in octaves. It is sometimes difficult to decide in which octave the pitch of a small bell lies, especially if the frequencies of these three partials are not exactly in 1:2:4 ratio. Difference tones between these partials may be heard.

Some authorities question the existence of subjective strike notes, and suggest that the strike note is an auditory illusion resulting from the nominal, the strongest partial during the first few seconds.²² Because of this, some English founders denote the pitch of the bell as the pitch of the nominal. There is some feeling that a bell rung in full circle sounds a different pitch from a bell struck somewhat more gently in a carillon.²³ Whether a founder tunes the nominal or the strike note makes little difference, however, because the nominal is one of the partials that determine the tuning of the strike note.

A handbell, unlike a church bell, appears to sound its fundamental pitch almost from the very onset of sound. There are several reasons for the absence of a separate strike note. First of all, there is no group of harmonic partials to create a strong subjective tone. Secondly, handbells employ a relatively soft non-metallic clapper, so that there is no sound of metal on metal, and there is a delay in hearing the partials after the clapper strikes the bell.

5.3. Sound radiation

The most prominent partials in the spectrum of a church bell or carillon bell are radiated by the ring-driven modes which some founders classify as belonging to group I. These modes can be considered to be due to standing flexural waves. With the exception of the (2, 0) mode (hum), all such modes have a nodal ring about half-way up the bell. For the purpose of understanding the general properties of the radiation field of the bell, we can model its outside surface as a collection of $4m$ piston-like sources alternating in phase ($2m = 4$ sources in the case of the (2, 0) mode). Other $4m$ sources of slightly smaller size occur on the inside surface.

The radiation efficiency of such a collection of alternating sources increases rapidly with frequency and with the size of the bell. As the bell increases in size, the area of each source increases, of course. But a more significant increase in radiation efficiency occurs when the separation between adjacent sources of opposite phase exceeds a half wavelength of sound in air. Another way to express this condition is that when the speed of flexural waves in the bell exceeds the speed of sound in air radiation efficiency increases markedly. The speed of flexural waves in a plate is given by $c_F = \sqrt{1.8c_L hf}$, where h is thickness, f is frequency, $c_L = \sqrt{E/\rho}$ is the longitudinal wave velocity, E is Young's modulus, and ρ is density.

The flexural wave speed is roughly the mode frequency times the circumference divided by m . For the church bell in Fig. 7, having a diameter of 70 cm, the flexural wave speed is roughly $(292.7)(0.7\pi/2) = 322 \text{ m s}^{-1}$ for the (2, 0) mode, but it increases to about 644 m s^{-1} for the (4, 1) mode and to about 1180 m s^{-1} for the (9, 1) mode. Since all but the lowest mode exceed the speed of sound in air (344 m s^{-1} at 23°C), the bell radiates most of its partials quite efficiently.

In a handbell, on the other hand, the walls are much thinner, and so the flexural wave speed is considerably smaller than in a heavy church bell. In the bell described in Fig. 11, for example, the flexural wave speed is roughly $(523)(0.12\pi) = 100 \text{ m s}^{-1}$ for the fundamental (2, 0) mode. Since this is considerably less than the speed of sound in air, the radiation is not very efficient. The (3, 0) and (4, 1[#]) modes in the same handbell have flexural wave speeds of approximately 200 m s^{-1} and 300 m s^{-1} , respectively, and thus these modes tend to radiate sound more efficiently than the fundamental (2, 0) mode. In large handbells, the radiation efficiency is low, because the flexural wave velocity for all the principal modes is less than the speed of sound in air. In a G_2 handbell, for example, the flexural wave speeds are only 41 m s^{-1} and 71 m s^{-1} for the (2, 0) and (3, 0) modes, respectively. This is a significant problem in handbell design.

In addition to the direct radiation of sound normal to its vibrating surfaces, a bell also radiates sound axially at twice the frequency of each vibrational mode.¹⁰ The intensity of this axially radiated sound increases with the fourth power of the vibrational amplitude, whereas the direct radiation increases only with the square of the amplitude.

The fundamental (2, 0) mode in a handbell radiates a fairly strong second harmonic partial along the axis as well as a fundamental whose maximum intensity is perpendicular to the axis. The (3, 0) mode also radiates at twice its vibrational frequency, but its partial is usually quite weak. The principal harmonic partials in the handbell sound are thus the first, second and third harmonics.¹⁰

5.4. Sound decay

A vibrating bell loses energy both by sound radiation and through internal losses.²⁴ Internal damping is normally larger in the low modes, but in the high modes radiation damping predominates. The sound pressure level of each radiated partial decays at a constant rate (i.e. the vibrational energy decays exponentially), and thus it is convenient to express the 60 dB decay time for each main partial.

The decay times of the principal modes of vibration of the bell described in Fig. 7 are given in Table 5. Note the long decay time of the (2, 0) mode (hum) and the relatively short decay times of the modes of higher frequency. This is mainly due to the greater radiation efficiency of the higher modes. Schad and Warlimont²⁵ found that the damping due to internal losses was approximately the same for all the principal modes, so the large differences in decay times are indicative of different rates of radiation.

The decay times for the tuned (2, 0) and (3, 0) modes in several handbells are given in Table 6. The general trend again is in the direction of decreasing decay times with increasing frequency. The second harmonic partial in each bell has half the decay time of the (2, 0) mode, because its intensity is proportional to the square of the amplitude of that mode.¹⁰

TABLE 6
Decay Times in Handbells of Various Sizes
(from Ref. 10)

<i>Bell</i>	<i>Decay times (s)</i>	
	<i>(2, 0) mode</i>	<i>(3, 0) mode</i>
G ₄	30	22
A ₄	34	20
B ₄	23	23
C ₅	20	20
D ₅	21	17
E ₅	20	17
F ₅	13	12
G ₅	20	14
A ₅	12	7.8
B ₅	12	7.4
C ₆	9	7.4
D ₆	6	5.5
E ₆	10	4.1
F ₆	6.6	2.4
G ₆	6.4	8.2

5.5. Warble

One of the prominent features in the sound of many bells is 'warble', which is caused by the beating together of the nearly degenerate components of a mode doublet.²⁶ Warble is observed as a variation in amplitude of the mean frequency at a rate determined by the frequency difference between the pair of modes. In theory, warble can be eliminated by selecting the strike point to lie at a node for one component and an antinode for the other. In practice this may not be useful because doublet splitting is likely to occur in many doublet pairs and selecting the strike point to minimize warble in one pair may enhance it in another. Various methods of trying to guarantee the correct alignment of doublet pairs by suitable breaking of the axial symmetry have been suggested, including for example the addition of two diametrically opposite meridian ribs at whose location the clapper strikes.²⁷

Warble is much easier to cope with in a handbell, because fewer modes of vibration contribute prominent partials to the bell sound. It is usually possible to select a strike point that reduces the warble in both the (2, 0) and (3, 0) modes to a tolerable level. The few bells in which this is not possible are usually scrapped by the founder.

5.6. Clappers

The sound of a bell is very much dependent on the size, shape, and hardness of the clapper, the point at which it strikes the bell, and the strength of the blow. Nevertheless, not much systematic research on clappers has been reported in the literature.

Church bell clappers were formerly made of wrought iron, which probably remains the preferred material, although in recent years it has been gradually replaced by spheroidal graphite cast iron and manganese brass. When a bell is rung 'full circle' (as in change ringing) the clapper strikes a powerful blow, and clappers must be carefully designed to avoid breaking. Clappers in carillon bells are usually made of manganese brass or steel.

Handbell clappers strike the bell with a soft surface of plastic, leather or felt. In some handbell clappers surfaces of different hardness can be selected in order to vary the timbre of the sound. Figure 14 shows some of the different timbres obtainable from three handbells by changing the hardness of the clapper surface. In general, the softer clapper favours the fundamental. Note the difference between sound radiated along the bell axis and that radiated at right angles to the bell.

Bigelow reports the results of an experiment in which an F_5 carillon bell

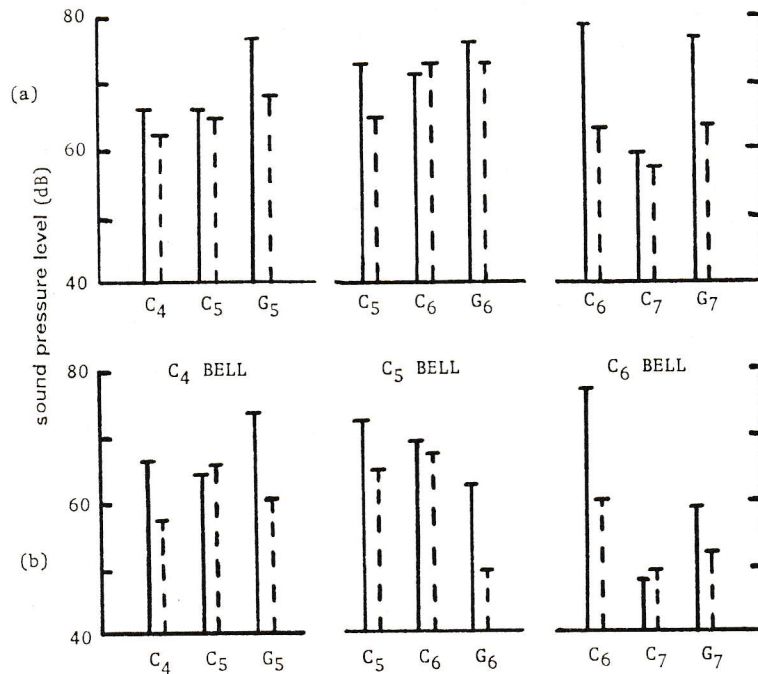


Fig. 14. Comparison of the principal partials of handbell sound radiated along the bell axis (a) and at right angles to the axis (b) with 'hard' (solid lines) and 'soft' (dash curve) clappers, respectively. Sound pressure levels were recorded 1 m from each handbell in an anechoic room (from Ref. 10).

was struck by three different clappers of varying weights each falling from three different heights.²⁸ A heavy hammer was found to increase the strength of the lower partials, but decrease those of the nominal (or octave). Since the nominal contributes in an essential way to the subjective strike note, it is implied that a heavy hammer will tend to diminish the intensity of that note. It should be noted that in a larger bell the nominal is usually found to be the most prominent partial for the first few seconds after the strike (see section 5.2).

6. SCALING OF BELLS

The fundamental frequency of a bell has roughly the same dependence on thickness h and diameter d as flexural vibrations in a circular plate: $f \propto h/d^2$. Thus it is possible to scale a set of bells by making all dimensions proportional to $1/f$. This scaling approximates that found in many carillons dating from the 15th and 16th centuries.²⁴ However, a $1/f$ scaling causes

the smaller treble bells to have a rather weak sound, and later bell founders increased the sizes of their treble bells.²⁸ The average product of frequency times diameter in several fine 17th-century Hemony carillons has been found to increase from 100 m s^{-1} (in bells of 30 kg or larger) to more than 150 m s^{-1} in small treble bells. The measurements of the bells in three Hemony carillons are shown in Fig. 15. The straight solid lines indicate a $1/f$ scale. Deviations from this scaling in the treble bells are apparent.

Studies of bells cast at the Eijsbouts bell foundry indicate that a $1/f$ scaling is used for swinging bells, but for carillon bells the diameter of the high-frequency bells is substantially greater than predicted by this scaling.²⁹ This and the consequent increase in thickness are probably the most notable differences between church bells and carillon bells.

The scaling of carillon bells can be conveniently described by writing the frequency of a bell as $f = k/r$, and the mass as $m = cr^3$, where r is the radius. Lehr²⁹ finds that it is the factor c/k that remains more or less constant as the size of carillon bells changes from small to large.

Scaling in handbells is generally less regular. The general scale followed

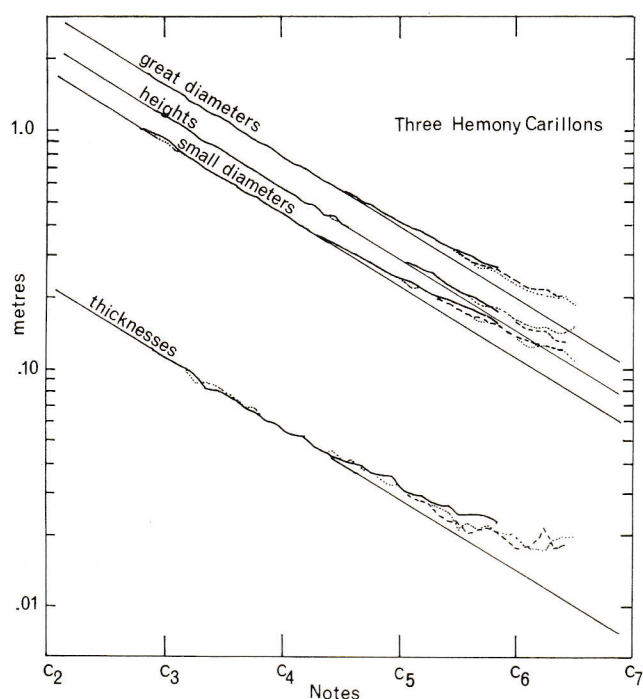


Fig. 15. Measurements of the bells in three Hemony carillons: — Oude Kerk, Amsterdam—Frans Hemony, 1658. ····· Dom Toren, Utrecht—Frans and Pieter Hemony, 1663. — Heilige Sulpiciuskerk, Diest—Pieter Hemony, 1670. The solid lines represent a $1/f$ scaling law (from Ref. 28).

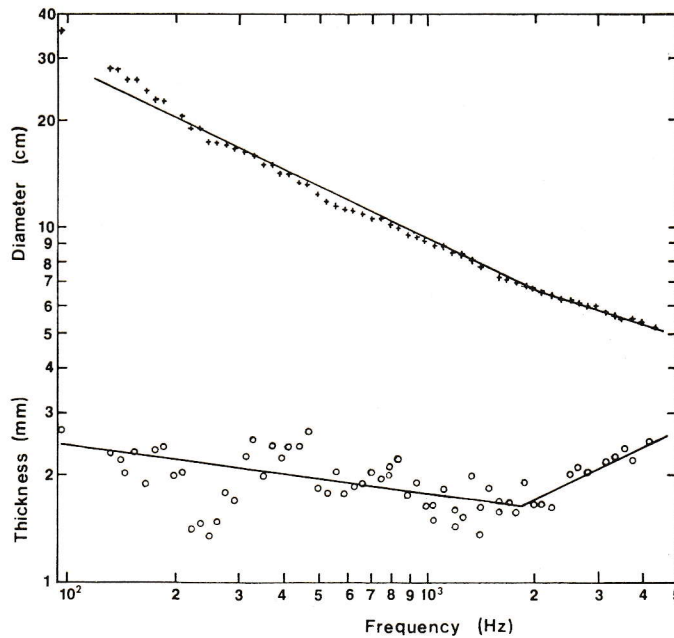


Fig. 16. Scaling of a five-octave set of Malmrk handbells (from Ref. 30).

by most bell crafters is to make the diameter inversely proportional to the square root of frequency, except for the smallest bells in which it varies inversely with the cube root of frequency instead.³⁰ The thickness is then adjusted, so that h/d^2 is nearly proportional to frequency. The scaling of a five-octave set of Malmrk handbells is shown in Fig. 16.

The various modal frequencies in church bells are found to have different dependences on thickness.³¹ In the modes of group I, which have antinodes near the soundbow, the frequency is roughly proportional to $h^{0.7}$, where h is thickness. In the modes of group II, which have antinodes near the waist, frequency is roughly proportional to $h^{0.86}$, which is nearer to the $f \propto h$ behaviour found in flat plates and circular cylinders.

7. CONCLUSION

Studies of the acoustical and vibrational behaviour of bells during the past hundred years have led to a pretty good understanding of how a bell vibrates and how it radiates sound. Recent studies have been much aided by the availability of digital computers and high-speed sound and vibration analysers. It is now possible to compute and to measure modal shapes to a high degree of precision.

Comparison of the modal shapes of the principal vibrational modes in

church bells and carillon bells with those observed in handbells reveals a number of interesting similarities as well as some important differences. In this paper, we have presented data in a way that facilitates these comparisons.

Inextensional modes in all three types of bells can be conveniently arranged in groups according to the numbers and locations of their nodal meridians and circles. Those modes that have no nodal circles (two or three modes in handbells but only one mode—the hum—in modern church and carillon bells) are assigned to Group 0. Group I modes have a nodal circle near the waist; Group II modes have a nodal circle near the mouth of the bell; the Groups III–IX have $n = 2$ –8 nodal circles at appropriate heights above the mouth. In no bell do we observe both a Group 0 and a Group II mode with the same number of nodal meridians. Ring axial, twisting and swinging modes have not yet been studied in handbells, nor have the various types of extensional modes that have been observed in church bells.

The markedly different timbres can be understood by examining the partials in the radiated sound from the different types of bells. The strike notes (subjective tones generated by three or more nearly harmonic partials) do not occur with handbells since these have only two harmonically tuned partials. A prominent octave partial, radiated indirectly by the first mode of vibration in a handbell, is not usually observed in church bells or carillon bells because of the strong octave partial radiated directly by the second mode of vibration. Sound radiation from the lowest modes of vibration in church bells and carillon bells is much stronger than in handbells because the speed of flexural waves in the thicker church bells exceeds the speed of sound in air whereas in the thinner handbells it does not.

Scaling follows rather different laws in the three types of bells. Large church bells and carillon bells tend to follow a $1/f$ scaling, but in most small carillon bells the dimensions decrease less rapidly than f^{-1} . In handbells the diameter tends to be proportional to $f^{-1/2}$ ($f^{-1/3}$ in the smallest bells), and the thickness is scaled accordingly.

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REFERENCES

1. P. Price, *Bells and man*, Oxford University Press, Oxford, 1983.
2. T. D. Rossing, *Acoustics of bells*, Van Nostrand Reinhold, Stroudsburg, Pa, 1984.
3. A. Lehr, *Van paardebel tot speelklok*, Europese Bibliotheek, Zaltbommel, The Netherlands, 1981.
4. R. Perrin and T. Charnley, Group theory and the bell, *J. Sound Vib.*, **31** (1973), pp. 411–18 (reprinted in Ref. 2).
5. F. G. Tyzzer, Characteristics of bell vibrations, *J. Franklin Inst.*, **210** (1930), pp. 55–6 (reprinted in Ref. 2).
6. R. Perrin, T. Charnley and J. de Pont, Normal modes of the modern English church bell, *J. Sound Vib.*, **90** (1983), pp. 29–49 (reprinted in Ref. 2).
7. A. Lehr, *Hedendaagse Nederlandse klokkengieterkunst* (Contemporary Dutch bell-founding art), Publ. No. 7, Netherlands Acoustical Society, 1965, pp. 20–49 (English translation in Ref. 2).
8. Lord Rayleigh (J. W. Strutt), *The theory of sound*, vol. 1, Macmillan, London, 1894 (reprinted by Dover, 1945).
9. T. Charnley, private communication, 1983.
10. T. D. Rossing and H. J. Sathoff, Modes of vibration and sound radiation from tuned handbells, *J. Acoust. Soc. Am.*, **68** (1980), pp. 1600–7.
11. R. L. Powell and K. A. Stetson, Interferometric vibration analysis by wavefront reconstruction, *J. Opt. Soc. Am.*, **55** (1965), pp. 1593–8.
12. R. N. Arnold and G. B. Warburton, Flexural vibrations of the walls of thin cylindrical shells having freely supported ends, *Proc. R. Soc. (Lond.)*, **A197** (1949), pp. 238–56.
13. M. Grützmacher, W. Kallenbach and E. Nellessen, Akustische Untersuchungen an Kirchenglocken, *Acustica* **16** (1965/66), pp. 34–45 (English translation in Ref. 2).
14. T. D. Rossing, Tuned handbells, church bells and carillon bells, *Overtones* **29** (1983) (1), pp. 15, 27–30; **29** (1983) (2), pp. 27, 29–30 (reprinted in Ref. 2).
15. E. F. F. Chladni, *Entdeckungen über die Theorie des Klanges* (Breitkopf and Härtel, Leipzig, 1787). (Translated excerpts in *Acoustics: historical and philosophical development*, ed. R. B. Lindsay, Dowden, Hutchinson and Ross, Stroudsburg, Pa, 1973).
16. T. D. Rossing, Chladni's law for vibrating plates, *Am. J. Phys.*, **50** (1982), pp. 271–4.
17. R. Perrin, T. Charnley, H. Banu and T. D. Rossing, Chladni's law and the modern English church bell, *J. Sound Vib.*, **102**(1) (1985), pp. 11–19.
18. T. D. Rossing, Acoustics of tuned handbells, *Overtones* **27**(1) (1981), pp. 4–10, 27.
19. F. H. Slaymaker, Chords from tones having stretched partials, *J. Acoust. Soc. Am.*, **47** (1970), pp. 1569–71.
20. J. F. Schouten and J. 'tHart, *De slagtoon van klokken* (The strike note of bells), Neth. Acoust. Soc. Publ. No. 7, (1965), pp. 8–19 (English translation in Ref. 2).
21. R. J. Ritsma, Frequencies dominant in the perception of the pitch of complex sounds, *J. Acoust. Soc. Am.*, **42** (1967), pp. 191–8.
22. M. J. Milsom, Tuning of bells, *The ringing world*, September 3 (1982), p. 733.

23. R. M. Ayres, Bell tuning—modern enigma or medieval mystery? *The ringing world*, September 23 (1983), p. 790.
24. E. W. van Heuven, *Acoustical measurements of church bells and carillons* (De Gebroeders van Cleef, 's-Gravenhage, 1949) (excerpts reprinted in Ref. 2).
25. C. R. Schad and H. Warlimont, Akustische Untersuchungen zum Einfluss des Werkstoffs auf der Klang von Glocken (Acoustical investigations of the influence of the material on the sound of bells), *Acustica*, **29** (1973), pp. 1–14 (English translation in Ref. 2).
26. R. Perrin and T. Charnley, The suppression of warble in bells, *Musical Instrument Technol.*, **3** (1978), pp. 109–17.
27. R. Perrin, T. Charnley and H. Banu, Increasing the lifetime of warble-suppressed bells, *J. Sound Vib.*, **80** (1982), pp. 298–303.
28. A. L. Bigelow, *The acoustically balanced carillon*, School of Engineering, Princeton University, 1961.
29. A. Lehr, A general bell formula, *Acustica*, **2** (1952), pp. 35–8.
30. H. J. Sathoff and T. D. Rossing, Scaling of handbells, *J. Acoust. Soc. Am.*, **73** (1983), pp. 2225–6.
31. A. Lehr, Partial groups in the bell sound, *J. Acoust. Soc. Am.*, **79** (1986), pp. 2000–11.
32. T. D. Rossing, Acoustics of bells, *Am. Scient.*, **72** (1984), pp. 440–7.
33. T. D. Rossing, R. Perrin, H. J. Sathoff and R. W. Peterson, Vibrational modes of a tuned handbell, *J. Acoust. Soc. Am.*, **76** (1984), pp. 1263–67.